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Fun with Functions

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DEPARTMENT OF MATHEMATICS
MIRANDA HOUSE
UNIVERSITY OF DELHI

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FUN WITH FUNCTIONS

1 Introduction

Functions are everywhere! We find functions almost everywhere be it computer systems, stock market, weather forecast, population forecast and many more. In this workshop, we study the concept of functions and understand them in detail with the help of graphs and various examples. Function is a special type of relation and the word 'function' is derived from a Latin word meaning 'operation'. Before studying the formal definition of a function let us first discuss about the relations.

2 Relation

A relation in mathematics, as the word suggests, defines a relationship between two non-empty sets. If there is a connection between the elements of two sets, then the two elements are said to be related to each other.

Definition 2.1 *Let P and Q be two sets. A relation from P into Q is a subset of $P \times Q$ (Cartesian product of P and Q).*

Let R be a relation from P into Q. If $(a, b) \in R$, we say that a is related to b with respect to R. We also write $(a, b) \in R$ as aRb.

Example 2.2 *Let $P = \{1, 2, 3, 4, 5\}$ and R be the relation given by $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$ which is a subset of $P \times P$. Here, we note that R is relating every element of P to itself, that is, aRa, for all $a \in P$.*

Example 2.3 *Let $P = \{1, 2, 3\}$ and $Q = \{a, b\}$. Then $R = \{(1, a), (1, b), (2, a)\}$ is a relation from P to Q.*

Example 2.4 *Let $A = \{2, 3, 5\}$, $B = \{4, 9, 25\}$ and $R = \{(a, b) \mid a \in A, b \in B, b = a^2\}$. Then R is a relation from A to B.*

3 Functions

Definition 3.1 *A function is a special type of relation from a non-empty set P into a non-empty set Q such that each element of P is related to a unique element of Q.*

A function from a non-empty set A to a set B is a rule which assigns a unique element of set B to each element of set A.

If f is a function from A to B, then we write $(a, b) \in f$ as $f(a) = b$ where $a \in A$ and $b \in B$. Here a is called the preimage of b under f and b is called the image of a under f.

A function f from A to B is denoted by $f : A \to B$. Here, the set A is called the **domain** of f and B is called the **codomain** of f. The set of all images of f is called the **range set** of f and is denoted by $f(A)$. Thus,

$$f(A) = \{f(a) \mid a \in A\}.$$

Example 3.2 *Let* $A = \{1, 2, 3, 4, 5\}$ *and* $B = \{2, 3, 5, 7, 9\}$. *Define* $f = \{(1, 2), (2, 3), (3, 5), (4, 9), (5, 3)\}$. *Then* f *is a function from* A *to* B *as every element of* A *is associated with unique element of* B. *Here* A *is domain,* B *is codomain and the set* $\{2, 3, 5, 9\}$ *is the range set of* f *respectively.*

Example 3.3 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined as* $f(x) = x$,*for every* $x \in \mathbb{R}$ *(*$\mathbb{R}$ *is the set of real numbers), that is,* f *is associating every element to itself. So* f *is a function from* $\mathbb{R}$ *to* $\mathbb{R}$ *having both domain and codomain as* $\mathbb{R}$. *Here we note that range set of* f *is also the set* $\mathbb{R}$.

Example 3.4 *Let* $A = \{1, 2, 3\}$ *and* $B = \{10, 11\}$ *be two sets. Let* $f_1 = \{(1, 10), (2, 10)\}$ *and* $f_2 = \{(1, 10), (1, 11), (3, 10), (2, 10)\}$. *Clearly,* f_1 *is not a function as* $3 \in A$, *does not have any image in* B *and* f_2 *is also not a function since* $1 \in A$ *has two images in* B *under* f_2.

Try to answer the following questions:

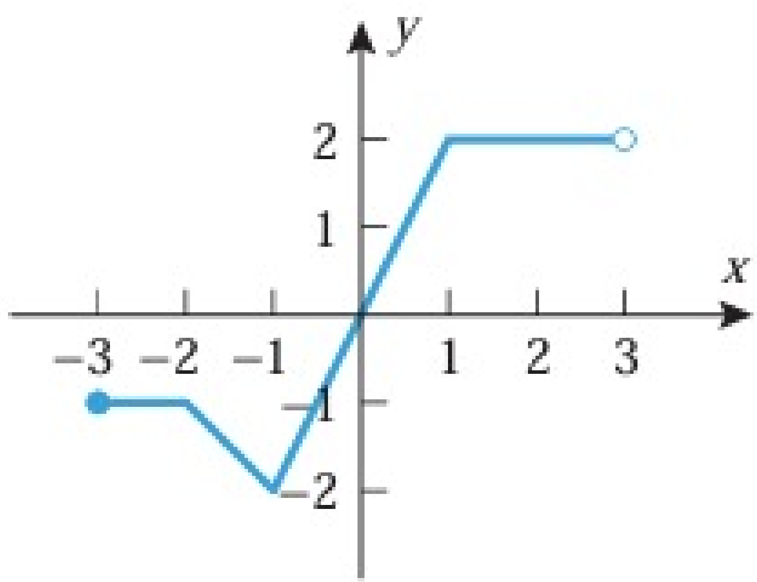
1. Which of the following relations is a function:

   (a) $f = \{(1, 1), (2, 1), (3, 1), (4, 1), (3, 3)\}$____________________

   (b) $f = \{(1, 2), (2, 3), (4, 2)\}$____________________

   (c) $f = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$____________________

2. The following figure shows the complete graph of $y = f(x)$.



   (a) The domain of f is ____________

   (b) The range of f is ____________________

   (c) $f(\frac{1}{2})$ =____________________

   (d) The solutions to $f(x) = \frac{-3}{2}$ are x =___________ and x =___________

3. Fill in the blanks:

Function	Domain	Range
$y = x^2$		
$y = \frac{1}{x}$		
$y = \sqrt{x}$		
$y = \sqrt{4 - x}$		
$y = \sqrt{1 - x^2}$		
$y = \sqrt{x^2 - 3x}$		
$y = \frac{1}{x-3}$		

4 Graph of a function

Definition 4.1 *Let f be a function from A to B, that is, $f : A \to B$. Then the graph of f is the set $\{(a, f(a)) \mid a \in A\}$ in $A \times B$.*

Vertical line test

A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.

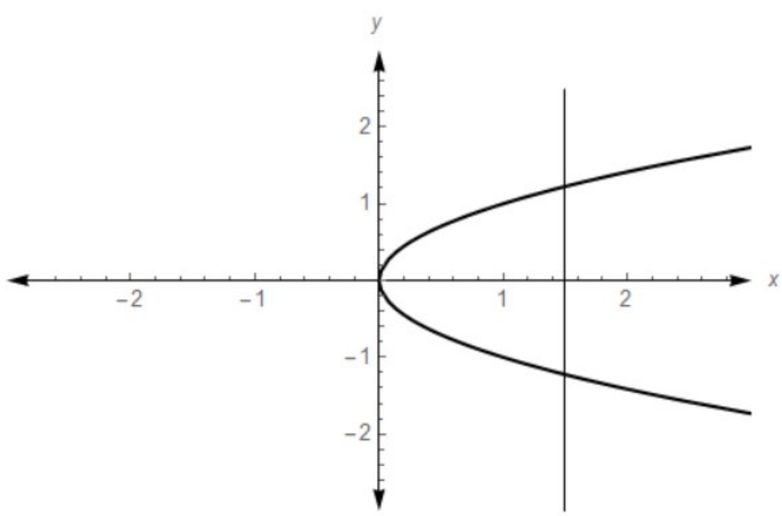
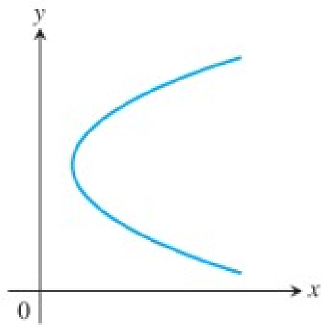


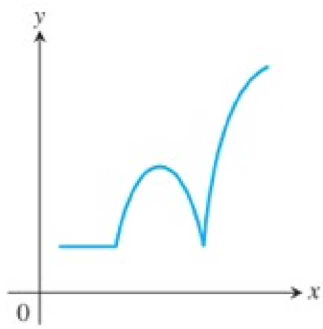
Figure 1: **Vertical Line Test (not a function)**

Try to answer the following question: Which of the graphs are graphs of functions of x and which are not?

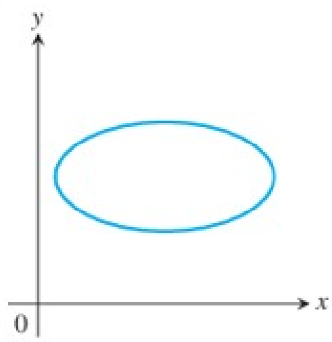
a.



b.



c.



We now illustrate the graphs of some basic functions through examples.

4.1 Constant Function

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 1$, for all $x \in \mathbb{R}$.

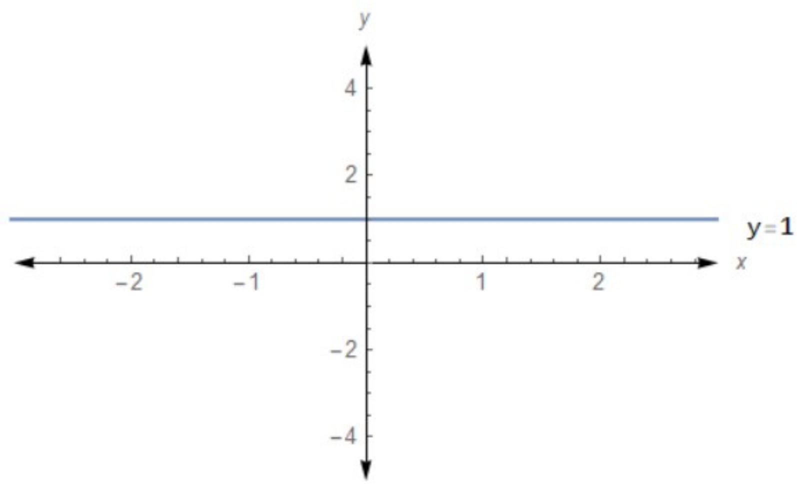


Figure 2: **Constant Function y=1**

Then graph of f is the set $\{(x, 1) \mid x \in \mathbb{R}\}$ is shown in figure-2.

4.2 Identity Function

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x$, for all $x \in \mathbb{R}$.

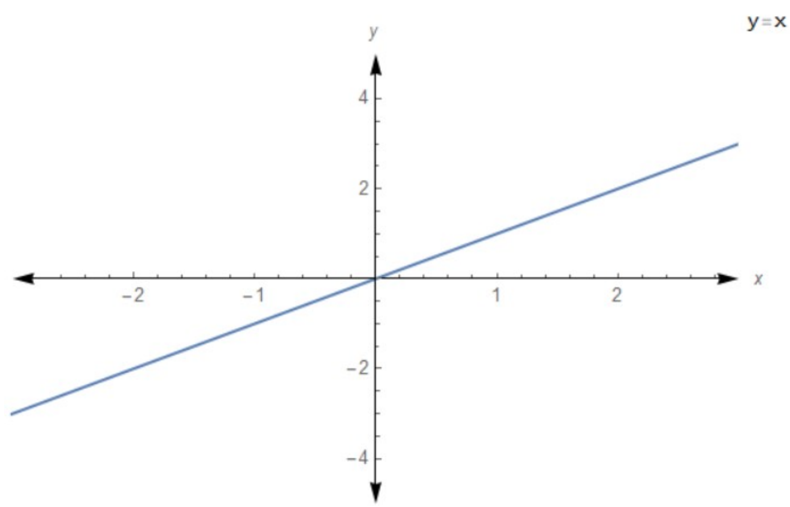


Figure 3: **Identity Function**

4.3 Power Function:

$$y = x^n,\ n \in \mathbb{N} \cup \{0\}$$

Example 4.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$, for all $x \in \mathbb{R}$.*

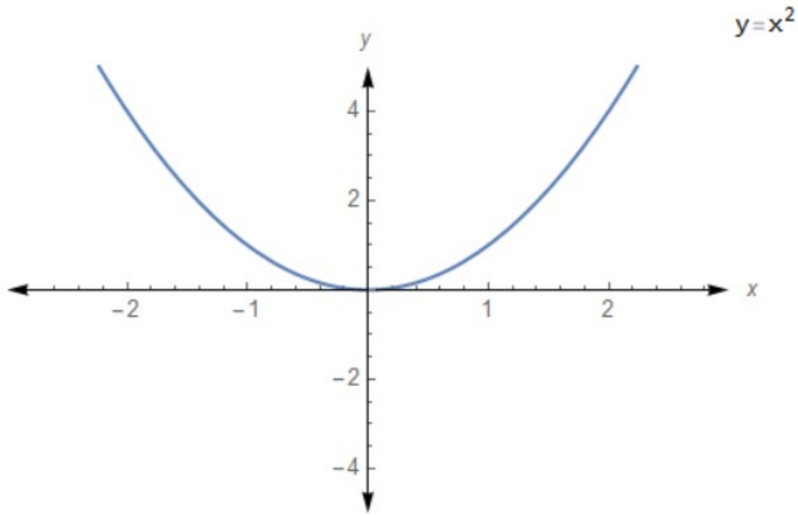


Figure 4: **Parabola**

Example 4.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^3$, for all $x \in \mathbb{R}$.*

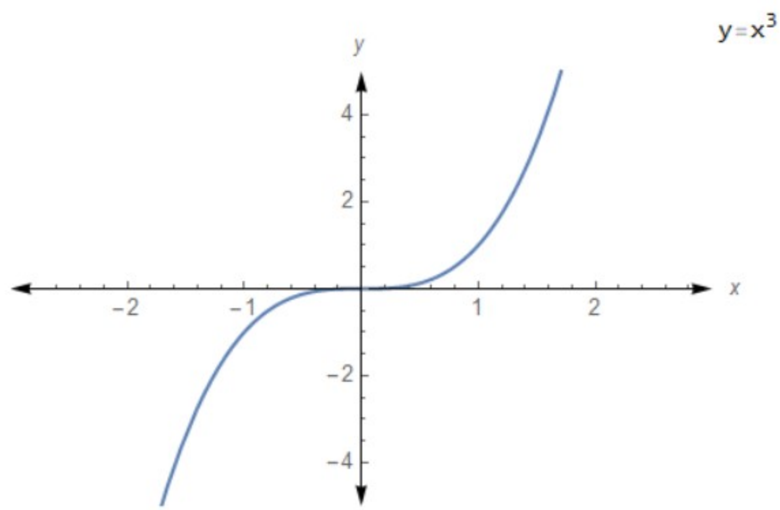


Figure 5: **Cubic Function**

Now, for even values of n, the functions $f(x) = x^n$ are even, so their graphs are symmetric about the y-axis. All the graphs have the general shape similar to $y = x^2$, and each graph passes through the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$. As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.

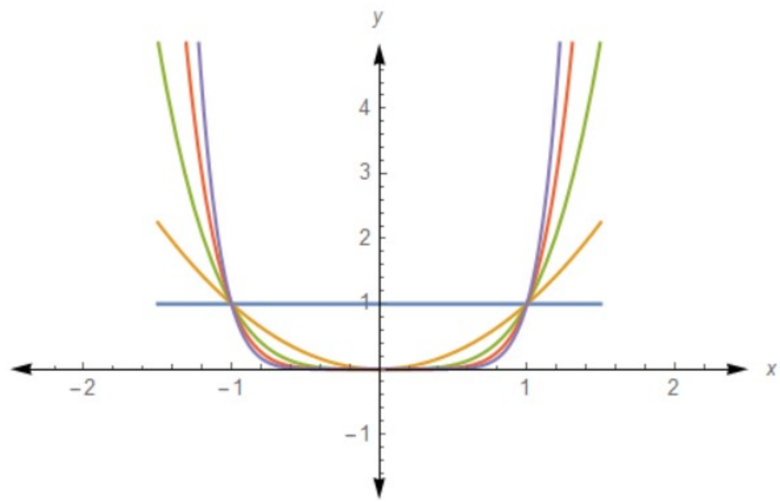


Figure 6: **Even Powered Functions**

For odd values of n, the functions $f(x) = x^n$ are odd, so their graphs are symmetric about the origin. All the graphs have the general shape similar to the curve $y = x^3$, and each graph passes through the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$. As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.

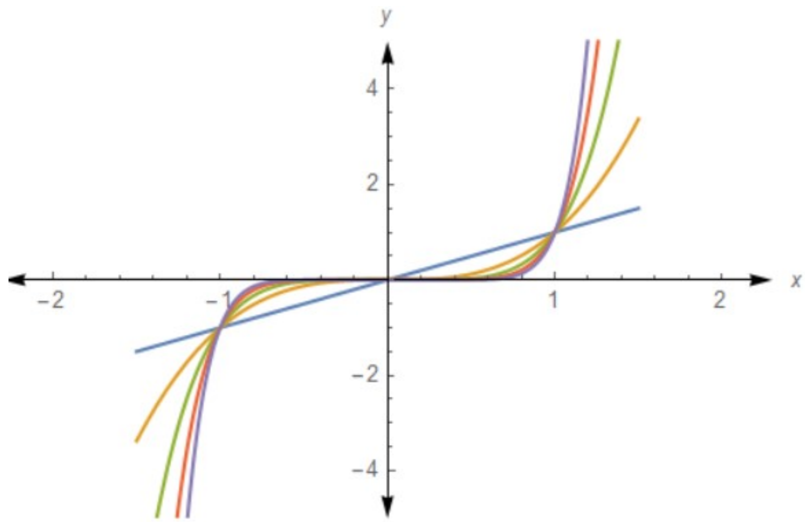


Figure 7: **Odd Powered Functions**

4.4 Piecewise Functions

Example 4.4 ***Absolute Value Function/ Modulus Function***

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

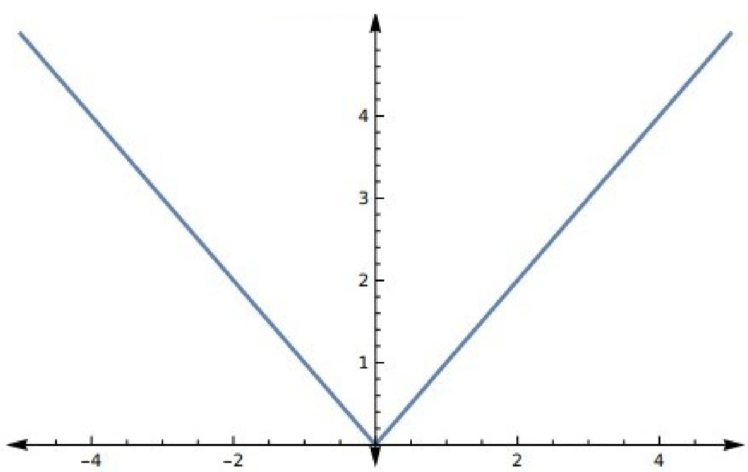


Figure 8: **Modulus function**

Example 4.5 ***Signum Function***

$$\begin{aligned} f(x) &= Sgn(x) \\ &= \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \\ &= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases} \end{aligned}$$

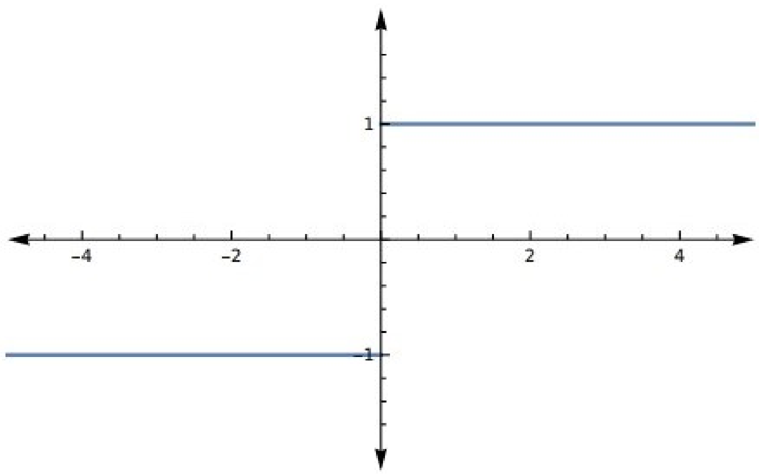


Figure 9: **Signum function**

Example 4.6 ***Greatest Integer Function***
$f(x) = \lfloor x \rfloor$ indicates the greatest integer less than or equal to x. Thus, $\lfloor 0.9 \rfloor = 0$, $\lfloor -1.2 \rfloor = -2$, $\lfloor 3.14 \rfloor = 3$ etc.

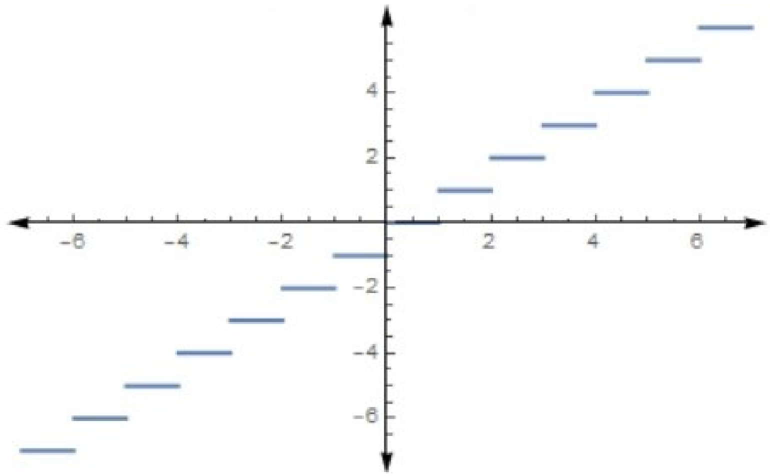


Figure 10: **Greatest integer function**

Example 4.7 ***Least Integer Function***
$f(x) = \lceil x \rceil$ indicates the smallest integer greater than or equal to x. Thus, $\lceil 0.53 \rceil = 1$, $\lceil -1.25 \rceil = -1$, $\lceil 2.89 \rceil = 3$ etc.

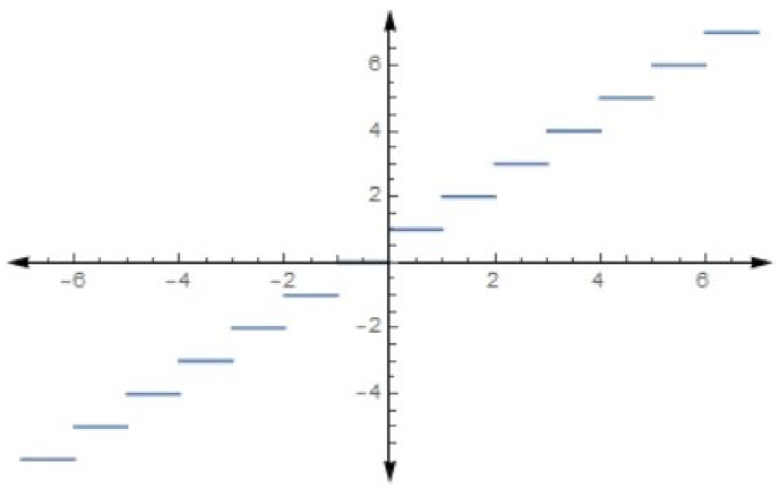


Figure 11: **Least integer function**

4.5 Trigonometric functions

Example 4.8 ***Sine Function***

$$f(x) = \sin x$$

Domain of sine function is $\mathbb{R}$ and its range is $[-1, 1]$. Sine function is periodic with period 2π.
Note: *A function $f(x)$ is periodic if there is a positive number p such that $f(x+p) = f(x)$, for all*

x. The smallest such value of p is called the period of f.

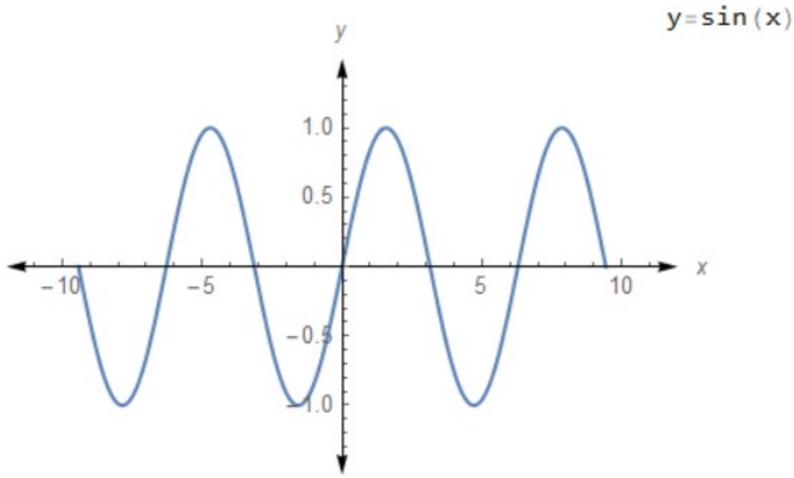


Figure 12: **sin x**

Example 4.9 ***Cosine Function***

$$f(x) = \cos x$$

Domain of cosine function is $\mathbb{R}$*, range is* $[-1, 1]$ *and is periodic with period* 2π*.*

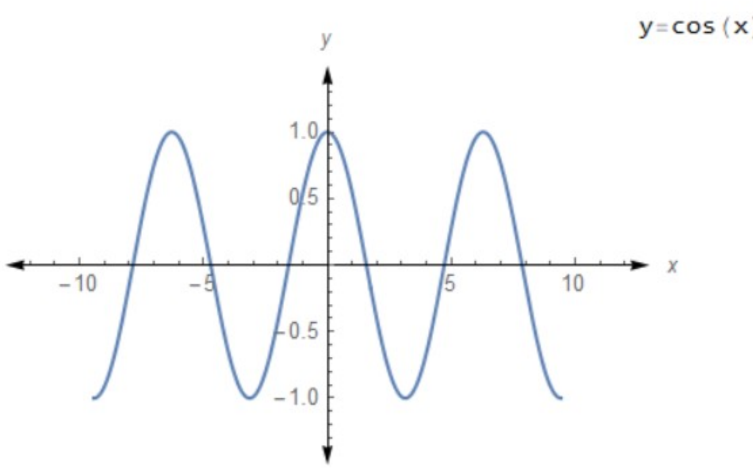


Figure 13: **cos x**

Example 4.10 ***Tangent Function***

$$f(x) = \tan x$$

Domain of tangent function is $\mathbb{R} \setminus \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$ *and its range is* $\mathbb{R}$*. It is a periodic function with period* π

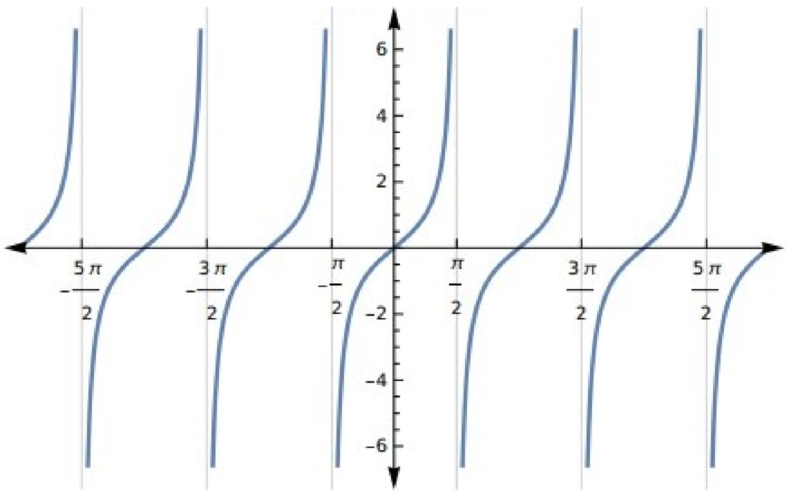


Figure 14: **tan x**

4.6 Exponential function:

$f(x) = e^x$ is called the natural exponential function. Domain of e^x is $\mathbb{R}$ and its range is the set of all positive real numbers, that is, $(0, \infty)$

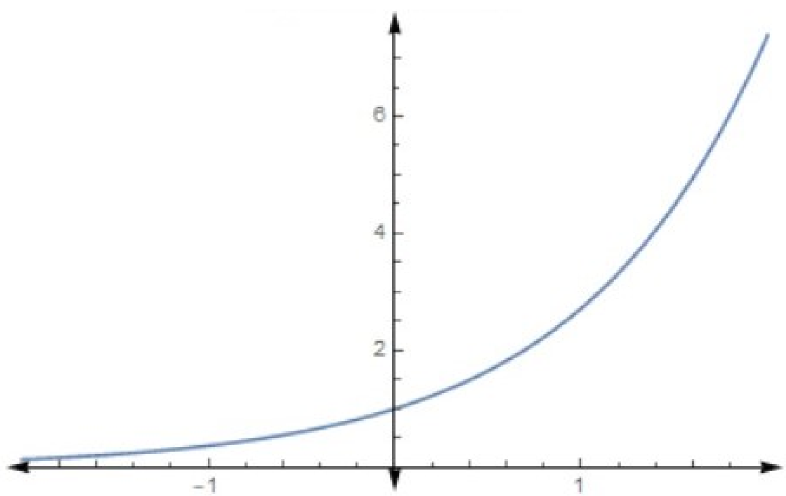


Figure 15: **Exponential Function**

4.7 Logarithmic function:

A function $f(x) = \log_e(x)$ is called the natural logarithm of x. Note that, $y = \log_e(x)$ if and only if $x = e^y$. Domain of log function is $(0, \infty)$ and its range is $\mathbb{R}$.

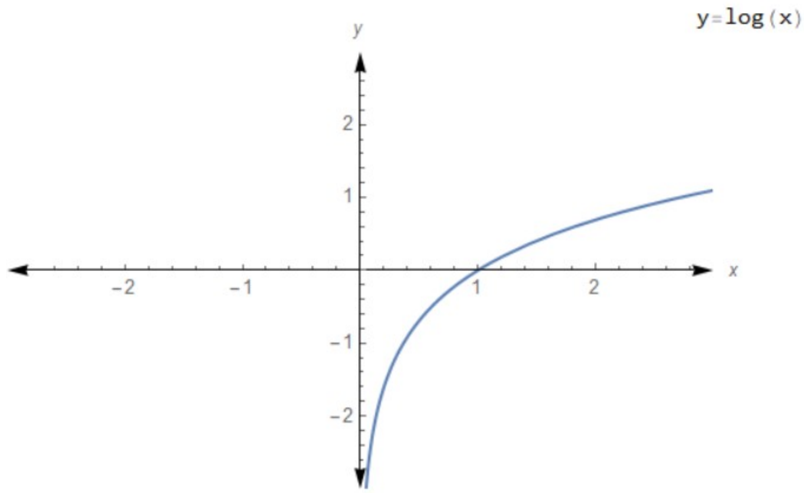


Figure 16: **Logarithmic Function**

5 Know your functions more

5.1 One-One and Onto Functions

A function $f : X \to Y$ is said to be a one-one(injective) function if the images of distinct elements of X under f are distinct. Thus, f is one -one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Some examples are x, x^3, $\log_e(x)$ etc.

A function $f : X \to Y$ is said to be an onto function if every element of Y is an image of some element in X under f, that is, for every $y \in Y$ there exists an element x in X such that $f(x) = y$. Some examples are x, x^3, $f(x) = e^x$ where $f : \mathbb{R} \to (0, \infty)$.

5.2 Even and Odd Functions

A function f is said to be an even function if $f(-x) = f(x)$ and is said to be an odd function if $f(-x) = -f(x)$. Geometrically, the graphs of even functions are symmetric about the y-axis because replacing x by $-x$ in the equation $y = f(x)$ yields $y = f(-x)$ which is same as the original equation $y = f(x)$. Likewise graphs of odd functions are symmetric about the origin. Some examples of even functions are x^{2n} $(n \in \mathbb{N})$, $\cos x$ and of odd functions are x^{2n-1} $(n \in \mathbb{N})$, $\sin x$.

5.3 Increasing and Decreasing Functions

If the graph of a function climbs or rises as we move from left to right, we say that the function is increasing. If the graph descends or falls as we move from left to right, the function is decreasing.

Below are the formal definitions.

Definition 5.1 *Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.*

(i) If $f(x_2) \leq f(x_1)$ whenever $x_2 < x_1$, then f is said to be increasing on I.

(ii) If $f(x_2) \geq f(x_1)$ whenever $x_2 < x_1$, then f is said to be decreasing on I.

Some examples of increasing functions are x^{2n-1} ($n \in \mathbb{N}$), greatest integer function, least integer function; and some examples of decreasing functions are $-x$, e^{-x}.